

NOWHERE MINIMAL CR SUBMANIFOLDS AND LEVI-FLAT HYPERSURFACES

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ABSTRACT. A local uniqueness property of holomorphic functions on real-analytic nowhere minimal CR submanifolds of higher codimension is investigated. A sufficient condition called almost minimality is given and studied. A weaker necessary condition, being contained a possibly singular real-analytic Levi-flat hypersurface is studied and characterized. This question is completely resolved for algebraic submanifolds of codimension 2 and a sufficient condition for noncontainment is given for non algebraic submanifolds. As a consequence, an example of a submanifold of codimension 2, not biholomorphically equivalent to an algebraic one, is given. We also investigate the structure of singularities of Levi-flat hypersurfaces.

1. INTRODUCTION

In this paper we investigate some local properties of nowhere minimal real-analytic CR submanifolds of higher codimension. In particular we are interested in a modulus uniqueness property for holomorphic functions, that is, when is a holomorphic function uniquely determined (up to a unimodular constant) by its modulus on a CR submanifold. We introduce a sufficient geometric condition called almost minimality, and we study related properties of such submanifolds. We also introduce a necessary condition, that is, being contained in a singular Levi-flat hypersurface, and thus we will find it necessary to study the structure of the singular set of such hypersurfaces.

Background material is taken mostly from [BER99]. We first fix some terminology. Let $M \subset \mathbb{C}^N$ be real-analytic submanifold defined near the origin. The tangent vectors of the form $\sum_{j=1}^N a_j \frac{\partial}{\partial \bar{z}_j}$ tangent to M are called the *CR vectors*. If this space has constant dimension on M , the submanifold is said to be a *CR submanifold*, and the complex dimension of the CR tangent space is called the *CR dimension* of M . If a CR submanifold is not contained in a proper complex analytic subvariety, we say it is a *generic submanifold*. We denote by Orb_p the *CR orbit* of M at p , that is, the germ of a CR submanifold of M through p of smallest dimension that has the same CR dimension as M . If $\text{Orb}_p = M$ as germs, then M is said to be minimal at p . If a real-analytic submanifold is minimal at one point, then it is minimal outside a real-analytic subvariety. If M is contained in a real-algebraic subvariety of \mathbb{C}^N of the same dimension as M , then M is said to be *real-algebraic*. We will say that a generic submanifold is *Levi-flat*, if there exist local holomorphic coordinates $z = (z_1, \dots, z_N)$, such that M can be given by $\text{Im } z_1 = \dots = \text{Im } z_d = 0$. A real-analytic, possibly singular hypersurface H (defined by the vanishing of a single real-analytic real valued function) is said to be *Levi-flat*, if near the nonsingular

points of hypersurface dimension, there exist local holomorphic coordinates, such that $\text{Im } z_1 = 0$ defines H . That is, Levi-flat hypersurfaces are locally (near nonsingular points) foliated by complex analytic hypersurfaces, and we call this the *Levi foliation*. We denote by H^* the nonsingular points of H that are of hypersurface dimension, that is, the points near which H is a real-analytic submanifold of real codimension 1. We define $H_s := H \setminus H^*$.

All minimal submanifolds have the modulus uniqueness property. The story is not so simple with nowhere minimal submanifolds. We introduce a sufficient geometric condition, called almost minimality, for a submanifold to have the modulus uniqueness property. For a generic submanifold M through the origin, M is almost minimal at 0, if for any connected neighbourhood U of 0, there exists a point $p \in U$, such that the CR orbit at p is not contained in a proper complex analytic subvariety of U . For example (see §8) the manifold M_λ , given by the defining equations

$$\begin{aligned}\bar{w}_1 &= e^{iz\bar{z}}w_1, \\ \bar{w}_2 &= e^{i\lambda z\bar{z}}w_2,\end{aligned}$$

is almost minimal at 0 if and only if λ is irrational.

The modulus uniqueness property described above is equivalent to the submanifold being locally contained in a possibly singular real-analytic Levi-flat hypersurface defined by the vanishing of the imaginary part of a meromorphic function. See §3 for discussion of the modulus uniqueness property. We will consider a weaker condition, that is, when the submanifold is contained in any Levi-flat hypersurface H . In particular, we will be concerned about when our higher codimension manifold M is contained in $\overline{H^*}$, which is not necessarily the same as H . Singular Levi-flat hypersurfaces with quadratic tangent cones have been studied by Burns and Gong in [BG99], and a similar approach, studying the Segre varieties of H , is taken in this paper. Burns and Gong also give an example of a Levi-flat hypersurface not defined by the vanishing of the real part of a meromorphic function. So being contained in a Levi-flat hypersurface is a potentially weaker condition on M than having the modulus uniqueness property. Singular Levi-flat hypersurfaces have also been studied by Bedford [Bed77] in case the singularity is contained in a codimension 2 complex variety.

For a generic submanifold $M \subset \mathbb{C}^N$, we will consider M in normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$, where d is the real codimension of M and n is the CR dimension of M , and M is given by $w = Q(z, \bar{z}, \bar{w})$, where Q is a holomorphic mapping defined in a neighbourhood of the origin in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d$, $Q(0, \zeta, \omega) \equiv Q(z, 0, \omega) \equiv \omega$, and $Q(z, \zeta, \bar{Q}(\zeta, z, w)) \equiv w$. We should note, however, that normal coordinates are not unique. The first main result of this paper is the following.

Theorem 1.1. *Let M be a germ of a generic real-analytic codimension 2 submanifold through the origin given in normal coordinates (z, w) and let M be nowhere minimal. Then $M \subset \overline{H^*}$, where H is a germ of a possibly singular real-analytic Levi-flat hypersurface if and only if the projection of M onto the second factor in (z, w) is contained in a germ of a possibly singular real-analytic hypersurface. If M is real-algebraic, then such an H always exists and is real-algebraic. Moreover, if M is not Levi-flat, then H is unique.*

Thus to answer the question of when M sits inside a Levi-flat hypersurface, it is sufficient to study the projection of M onto the second factor in normal coordinates,

if M is not real-algebraic. The question is fully answered in case M is real-algebraic. In fact, if such an H exists, then there exists one defined by an equation independent of z . Further, we will show that if two holomorphic functions f and g have equal modulus on M , then f/g also only depends on w .

We also prove that an almost minimal submanifold of codimension 2 cannot be contained in a Levi-flat hypersurface. So an almost minimal submanifold that is nowhere minimal is an example of a submanifold, which is not locally biholomorphic to a real-algebraic submanifold. For λ irrational, the submanifolds M_λ defined above are such examples. Examples of such hypersurfaces both minimal and nonminimal can be found in [BER00] and [HJY01].

When M is almost minimal at p , we will study the dimension of $\text{hol}(M, p)$, the space of *infinitesimal holomorphisms* at p , that is, the Lie algebra generated by germs at p of real-analytic vector fields X on M defined in some neighbourhood U of p , such that for each $q \in U$ there is another neighbourhood $q \in V \subset U$ such that the map $z \mapsto \exp tX \cdot z$ for $|t| \leq \epsilon$ is a CR diffeomorphism of M (a diffeomorphism of M that preserves the CR vector bundle).

A vector field X in \mathbb{C}^N is called a *holomorphic vector field*, if we can write it locally as $X = \sum_{k=1}^N a_k(z) \frac{\partial}{\partial z_k}$, where the a_k are holomorphic in $z \in \mathbb{C}^N$. A submanifold M is said to be *holomorphically nondegenerate* at $p \in M$, if there does not exist any germ at p of a nonzero holomorphic vector field tangent to M . If M is connected, real-analytic and generic it turns out, that if it is holomorphically nondegenerate at one point it is so at all points. Being holomorphically nondegenerate is a necessary condition for $\dim_{\mathbb{R}} \text{hol}(M, p) < \infty$. In the case M is a hypersurface Staton [Sta96] proved that this is in fact a sufficient condition. For higher codimension submanifolds, Baouendi, Ebenfelt and Rothschild [BER98] proved that $\dim_{\mathbb{R}} \text{hol}(M, p) < \infty$ if M is minimal at p , and if M is nowhere minimal, then on a dense open subset of M , $\dim_{\mathbb{R}} \text{hol}(M, p)$ is either zero or infinite. We prove the following result for almost minimal submanifolds.

Theorem 1.2. *Let $M \subset \mathbb{C}^N$ be a connected, real-analytic holomorphically nondegenerate generic submanifold and suppose $p \in M$ and M is almost minimal at p . Then*

$$\dim_{\mathbb{R}} \text{hol}(M, p) < \infty.$$

Finally, it will be necessary to know something about the structure of the singular set of a Levi-flat hypersurface. This result is also of interest on its own. We prove a technical theorem in §2 which has the following corollary.

Theorem 1.3. *Let $H \subset \mathbb{C}^N$ be a singular real-analytic Levi-flat hypersurface, and let $M \subset H_s \cap \overline{H}^*$ be a smooth submanifold. Then for p on an open dense set of M , the germ of M at p is contained in some germ of a complex variety or generic real-analytic Levi-flat submanifold of real dimension $2N - 2$.*

The paper has the following organization. In §2 we study the singular set of Levi-flat hypersurfaces and prove Theorem 1.3. In §3 we study the modulus uniqueness property. In §4 we consider when M is contained in a Levi-flat hypersurface and prove the first part of Theorem 1.1. In §5 we define and study the almost minimality condition and in §6 we prove Theorem 1.2. In §7 we study real-algebraic submanifolds and prove the remainder of Theorem 1.1. Finally in §8 we work out the example M_λ family of submanifolds and prove a slightly more general result which can be used for generating further examples of almost minimal submanifolds.

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2. SINGULARITY OF LEVI-FLAT HYPERSURFACES

As we noted before we will consider a singular real-analytic Levi-flat hypersurface through the origin $H \subset U$ where U is an open neighbourhood of the origin in \mathbb{C}^N given by a $H = \{z \in U \mid \rho(z, \bar{z}) = 0\}$ for a real valued real-analytic function ρ . As we are interested in local properties of H we will assume that U is small enough such that ρ can be complexified to $U \times {}^*U$ where ${}^*U = \{z \mid \bar{z} \in U\}$. Further, we will assume that U is connected. As before we will denote by H^* the nonsingular points of dimension $2N - 1$. Then we let $H_s := H \setminus H^*$. We note that it is not necessarily true that $\overline{H^*} = H$, even if H is irreducible. We say that H is Levi-flat, if near each $p \in H^*$ there are suitable holomorphic coordinates such that H is given by $\text{Im } z_1 = 0$. Burns and Gong [BG99] prove the following useful lemma.

Lemma 2.1. *Let H be an irreducible singular real-analytic hypersurface. Then if H is Levi-flat at a single point of H^* , then it is Levi-flat at all points of H^* .*

Our main result about Levi flat hypersurfaces is the following theorem.

Theorem 2.2. *Let $H \subset U \subset \mathbb{C}^N$ be a singular real-analytic Levi-flat hypersurface. Then*

$$H_s \cap \overline{H^*} \subset \bigcup_{j=1}^{\infty} M_j$$

where $M_j \subset U_j$ for some countable collection of open sets $U_j \subset U$, and where M_j is either a proper complex analytic subvariety of U_j or a generic real-analytic Levi-flat submanifold of real dimension at most $2N - 2$.

Theorem 1.3 in the introduction follows from this technical result.

Proof of Theorem 1.3. By the above theorem, $M \subset \bigcup M_j$. Suppose that there is no point in M such that near that point $M \subset M_j$ (as germs) for some j . That means, $M \cap M_j$ is nowhere dense in M (it does not contain an open set). But there are only countably many such sets, and so by Baire category theorem they cannot cover all of M , which would be a contradiction. Thus there has to exist a point p where M is contained (as a germ at p) in some M_j . This holds on an open set near p as well, and furthermore, since it holds for all open U , by taking U smaller we can see that it has to hold on an open dense set of M . \square

A useful weaker result, at a point where H_s is a submanifold of codimension one in H , is the following.

Corollary 2.3. *Let H be a singular real-analytic Levi-flat hypersurface defined in a neighbourhood of the origin in \mathbb{C}^N , and suppose that H_s is a manifold of dimension $2N - 2$ and $H_s \subset \overline{H^*}$. Then H_s is either complex analytic or Levi-flat.*

Proof. If H_s was of a different type, then all the Levi-flat and complex analytic M_j 's have an intersection of a lower dimension with H_s . By Baire category theorem again, this is not possible, as there are only countably many. \square

Thus we have a complete categorization of singularities if they are of highest possible dimension and are in the closure of the nonsingular points. There are examples where the singular set is complex (e.g. $\{z \mid \operatorname{Im} z_1^2 = 0\}$) or Levi-flat (e.g. $\{z \mid (\operatorname{Im} z_1)(\operatorname{Im} z_2) = 0\}$), but it is not clear that an irreducible hypersurface can have a Levi-flat singularity.

A smooth CR submanifold is said to be of *finite type* at $p \in M$ if the CR vector fields, their complex conjugates and finitely many commutators, span the complexified tangent space $\mathbb{C}T_p M$ ($\mathbb{C}T_p M = \mathbb{C} \otimes_{\mathbb{R}} T_p M$). In case M is real-analytic, being finite type at p is equivalent to being minimal at p . It is not hard to see that if M is finite type at p , then there cannot exist a holomorphic function in a neighbourhood of p which is real valued on M . We can now rule out all smooth finite type generic submanifolds of any codimension being contained in Levi-flat hypersurfaces.

Corollary 2.4. *Let $H \subset U \subset \mathbb{C}^N$ be singular real-analytic Levi-flat hypersurface, and let $M \subset \overline{H}^*$ be a smooth generic submanifold. Then M is not of finite type at any point.*

Proof. Take a point $p \in M$. If $p \in M \cap H^*$, then there is some neighbourhood of p , where in suitable local coordinates H is given by $\operatorname{Im} z_1 = 0$, and thus z_1 is real valued on an open set of M . Since if M would be of finite type at p , it would be of finite type in a neighbourhood of p . If there exists a real valued holomorphic function on M near p , M cannot be of finite type at p . So let $p \in H_s$. Again if M would be of finite type at p then it would be so near p , and there would either be a point $q \in M \cap H^*$ where M was of finite type, which we now know cannot happen, or $M \subset H_s$ as germs at p . But then by Theorem 1.3 for some point $q \in M$, where M would be of finite type, it would be contained as germ in either a complex variety or a Levi-flat generic submanifold which is again impossible. Thus M cannot be of finite type. \square

Before going into the proof of Theorem 2.2, let's fix some notation and background. Let Σ_z be the *Segre variety* of H at the point z , that is the set $\{\zeta \in U \mid \rho(\zeta, \bar{z}) = 0\}$, and let Σ'_z be the branches of Σ_z completely inside H . We say that Σ_z is *degenerate* if Σ_z contains an open set of \mathbb{C}^N , that is, if $\Sigma_z = U$ if U is connected.

We will need some lemmas about Levi-flat hypersurfaces. Both of the following are given (in more generality) and proved in [BG99].

Lemma 2.5. *If ρ is an irreducible germ of a real-analytic function near 0 in \mathbb{C}^N , and $H := \{z \mid \rho(z, \bar{z}) = 0\}$ has dimension $2N - 1$, then for any neighbourhood U of 0, there is a smaller neighbourhood $U' \subset U$ of 0, such that if $\hat{\rho}$ is any real-analytic function on U which vanishes on an open set of $H^* \cap U'$, then ρ divides $\hat{\rho}$ on U' . Further, ρ is irreducible as a germ of a holomorphic function near origin in \mathbb{C}^{2N} .*

Lemma 2.6. *Let $H \subset U$ be as above and Levi-flat, and suppose $z \in H$ is such that Σ_z is non-degenerate. Then Σ'_z is non-empty, and further one branch of Σ'_z passes through z . If $z \in H^*$, then Σ'_z has only one branch through z , and this is the unique germ of a complex variety through z .*

Also, since we could pick U smaller and smaller, one branch of Σ'_z must therefore always pass through z .

If ρ is a defining function for H in a neighbourhood U , then at all points of H_s , ρ must have a vanishing gradient, since otherwise H would be a nonsingular hypersurface at that point. In fact, picking a possibly smaller U , $\{z \in H \mid \partial\rho(z, \bar{z}) = 0\}$ is a proper subvariety of H containing H_s (here ∂ means the exterior derivative in the z variables). Assume H is irreducible, complexify ρ into $U \times {}^*U$, and let $\mathcal{H} = \{(z, \zeta) \in U \times {}^*U \mid \rho(z, \zeta) = 0\}$. Then by Lemma 2.5, ρ is irreducible as a holomorphic function (in a possibly smaller neighbourhood), and thus generates the ideal of \mathcal{H} by the Nullstellensatz at every point in $U \times {}^*U$. Therefore, the gradient of the complexified ρ does not vanish at all nonsingular points of \mathcal{H} . Near any $p \in H^*$ we have a local defining function with nonvanishing gradient near p , which when complexified divides ρ . That means, near p , H^* complexifies to a germ of a smooth complex hypersurface in $U \times {}^*U$ contained in \mathcal{H} . Since H^* is totally real in this complex hypersurface we know $\partial\rho$ cannot vanish identically on H^* (or it would vanish in all of \mathcal{H} since it is irreducible). Hence, $\partial\rho = 0$ defines a proper lower dimensional subvariety of H which contains H_s . We can't quite say it equals H_s , as a point p could be in H^* , but the point (p, \bar{p}) could a priori be a singular point of \mathcal{H} .

Lemma 2.7. *Let $H_1, H_2 \subset \mathbb{C}^N$ be two connected nonsingular real-analytic Levi-flat hypersurfaces, such that $0 \in H_1 \cap H_2$. If U is a sufficiently small neighbourhood of 0 , and $H_1 \cap U \neq H_2 \cap U$, then there exists a possibly empty proper complex analytic subvariety $A \subset U$ such that $(U \cap H_1 \cap H_2) \setminus A$ is either empty or a generic real-analytic Levi-flat submanifold of codimension 2.*

Proof. We let U be small enough such that $H_1 \cap U$ and $H_2 \cap U$ are closed in U and hence we can assume that $H_1, H_2 \subset U$. Further, let U be small enough such that there exist holomorphic coordinates in U where H_1 is given by $\text{Im } z_1 = 0$ and H_2 is given by $\text{Im } f = 0$, where f is holomorphic with nonvanishing differential. The set where the complex differentials of f and z_1 are linearly dependent is a complex analytic subvariety. If the complex differentials are everywhere linearly dependent then f depends only on z_1 and thus the intersection of H_1 and H_2 is complex analytic. So suppose that outside a subvariety A , f and z_1 have linearly independent differentials so locally in an even smaller neighbourhood we can change coordinates again to make $f = z_2$ and then the intersection is locally defined as $\text{Im } z_1 = \text{Im } z_2 = 0$ and we are done. \square

Proof of Theorem 2.2. Recall that to prove the Theorem, we will cover $H_s \cap \overline{H^*}$ by countably many Levi-flat submanifolds of codimension 2 and local complex analytic subvarieties. These submanifolds and subvarieties need not lie in H itself, we just want their union as sets to contain $H_s \cap \overline{H^*}$.

Let $H'_s := H_s \cap \overline{H^*}$. The place in the proof where we fail to cover all of H_s , if $H_s \not\subset \overline{H^*}$, is in the application of Lemma 2.6.

Assume that H is irreducible. If it is reducible, and we prove the result for each branch, then it is also true for the union of those branches. This is because if K and L are branches of $H = K \cup L$, then $H_s = K_s \cup L_s \cup S$, where S is the set of points of $K^* \cap L^*$, where $K^* \cap L^*$ is not a hypersurface. Hence, if we have covered K_s and L_s , the only other points that need to be covered are points of S . If $p \in S$ we pick a small enough neighbourhood of p and apply Lemma 2.7. We can also

assume H it is irreducible in arbitrarily small neighbourhoods of 0 as well for the same reason (so irreducible as a germ).

First we note that the points $z \in U$ where Σ_z is degenerate are inside a complex analytic variety, because $z \in \Sigma_w$ implies by reality of ρ that $w \in \Sigma_z$. So that means that if z is a degenerate point, then it is contained in Σ_w for all $w \in U$, and thus is inside a complex analytic subvariety A . Because we only care about a countable union of local varieties and manifolds, we can just cover $U \setminus A$ by smaller neighbourhoods and work there. Thus we can assume that U contains no degenerate points.

Suppose $0 \in H$, and suppose that a branch of Σ_0 , call it A again, is contained in H_s . Again, since we only care about a countable union of local varieties and manifolds, we can cover $U \setminus A$ by small neighbourhoods and work there. Thus we can assume that Σ_0 has no branch that is contained in H_s (and thus not in H'_s).

By Lemma 2.6, Σ'_0 is non-empty and we now know that no branch of it is contained completely in H'_s . So we know that there exists a point $\zeta \in \Sigma'_0$ such that $\zeta \in H^*$. As Σ'_0 at $\zeta \in H^*$ is the unique complex variety (again by Lemma 2.6) passing through ζ we know that Σ'_ζ shares this branch with Σ'_0 .

We can of course pick this ζ in a topological component of $(\Sigma'_0)^* \cap H^*$, where $(\Sigma'_0)^*$ is the nonsingular part of Σ'_0 , such that 0 is in the closure of this component. As no branch of Σ'_0 lies inside H'_s and there is at least one branch through 0, then at least one topological component of $(\Sigma'_0)^* \cap H^*$ will be such that 0 is in its closure.

We look at a small neighbourhood V of ζ such that $H \cap V$ is connected and nonsingular, and further, such that H is defined in V by $\text{Im } f(z) = 0$, for some f holomorphic in V where the gradient of f does not vanish in V .

Pick a nonsingular real-analytic curve $\gamma: (-\epsilon, \epsilon) \rightarrow H$ such that $\gamma(0) = \zeta$, $\{\gamma\} \subset V$, and furthermore, that γ is transverse to the Levi foliation of H^* . We can do this by just changing coordinates in V such that $z_n = f$, and then our curve might be $t \mapsto t\alpha$ where $\alpha \in C^n$ and α_n is not real. Once we have γ we can look at the sets $\Sigma_{\gamma(t)}$ for various t . These are given by $\{z \mid \rho(z, \overline{\gamma(t)}) = 0\}$. However, we can just look at the zero set of the function $(z, t) \mapsto \rho(z, \overline{\gamma(t)})$ as t is real. Further, we can pick γ such that $\rho(0, \overline{\gamma(t)})$ is not identically zero since if it were for all choices of γ (by varying α above), then Σ_0 would contain an open set in H^* and thus would be degenerate, and we assumed it was not. We can complexify t and look at the zero set of $\rho(z, \overline{\gamma(t)})$ in $U \times D_\epsilon$ (where D_ϵ is the disk of radius $\epsilon > 0$).

Next apply the Weierstrass preparation theorem, which we can do in some neighbourhood of $(0, 0)$ in $U' \times D_{\epsilon'} \subset U \times D_\epsilon$ and we get a polynomial

$$F(z, t) = t^m + \sum_{j=0}^{m-1} a_j(z) t^j,$$

whose zero set is the zero set of $\rho(z, \overline{\gamma(t)})$. Outside of the discriminant set of F , $\Delta \subset U'$, we have (locally) m holomorphic functions $\{e_j\}_1^m$ which give us the solutions to $F(z, e_j(z)) = 0$. We look at the places where these solutions are real, that is the points in U' where $e_j - \bar{e}_j = 0$. To be able to complexify we look at the function

$$i^m \prod_{j,k=1}^m (e_j(z) - \overline{e_k(z)}).$$

It is easy to see that $\overline{e_j(z)}$ is a real function. Furthermore, it is symmetric both in the $e_j(z)$ and the $e_k(z)$, this means that after complexification we have a well defined holomorphic function in $(U' \times {}^*U') \setminus (\Delta \times {}^*\Delta)$, and continuous in all of $(U' \times {}^*U')$ and thus holomorphic in $(U' \times {}^*U')$. (see [Whi72] for more). Thus we have a real-analytic function, say $\hat{\rho}: U' \rightarrow \mathbb{R}$ that is locally outside of Δ given by $i^m \prod_{j,k=1}^m (e_j(z) - \overline{e_k(z)})$.

We let $\hat{H} := \{\hat{\rho} = 0\}$. We need to now see that $(H \cap U') \cap \hat{H}$ is open in H , because then $(H \cap U') \subset \hat{H}$ as H is irreducible in U' and we can apply Lemma 2.5 as we can always pick a smaller U' .

It is obvious that $\Sigma'_{\gamma(0)} \cap U'$ is in both H and \hat{H} . The trouble is for other t , as $V \cap U'$ may in fact be empty. Because of how we picked ζ , we note that the topological component of $(\Sigma'_{\gamma(0)})^*$ where ζ lies is connected to 0. So we can find a nonsingular point ζ' of $\Sigma'_{\gamma(0)}$ on this component that is arbitrarily close to 0, and thus inside U' . We can pick a finite sequence of overlapping neighbourhoods $\{V_j\}$ from ζ to ζ' such that inside each V_j , H is given by $\text{Im } f_j(z) = 0$ (for some f_j holomorphic in V_j). We call the final neighbourhood V' and assume $V' \subset U'$ and there H is given by $\text{Im } f'(z) = 0$ (for some f' holomorphic in V'). It is easy to see that the Levi foliation is given by $f_j(z) = r$ for some real r , and that these sets must agree on $V_j \cap V_k$. Thus for some $\epsilon'' > 0$, for all $|t| < \epsilon''$, we have a component of $\Sigma_{\gamma t}$ passing through V which also passes thorough V' which contains ζ' . But $V' \subset U'$ and \hat{H} and H both contain all points $\{z \mid f'(z) = t, |t| < \epsilon''\}$ and that is an open set in H .

Now that we know that H is contained in \hat{H} we can remove Δ which is complex analytic and work only in small neighbourhoods where \hat{H} is given by $i^m \prod (e_j(z) - \overline{e_k(z)})$. Since $e_j(z) - \overline{e_k(z)}$ is pluriharmonic, and thus its real and imaginary parts are pluriharmonic, meaning that we can represent them as the imaginary part of a holomorphic function, that is $\text{Im } f_{jk}(z) + i \text{Im } g_{jk}(z)$. Thus we get locally that

$$\hat{\rho}(z, \bar{z}) = i^m \prod_{j,k=1}^m (\text{Im } f_{jk}(z) + i \text{Im } g_{jk}(z)).$$

If $\text{Im } f_{jk}(z) + i \text{Im } g_{jk}(z)$ is zero then $(\text{Im } f_{jk}(z))(\text{Im } g_{jk}(z))$ is also zero. Thus we can make yet a larger surface by looking at the zero set of

$$\hat{\hat{\rho}}(z, \bar{z}) = \prod_{j,k=1}^m (\text{Im } f_{jk}(z))(\text{Im } g_{jk}(z)).$$

That is just a product of the imaginary parts of holomorphic functions. We can now take out the set where the gradient of f_{jk} and g_{jk} vanish, which is a complex analytic set and work in smaller neighbourhoods outside this set. We can take these neighbourhoods small enough such that each $\text{Im } f_{jk} = 0$ and $\text{Im } g_{jk} = 0$ defines a nonsingular, connected hypersurface. The singular set of H must be contained in the intersection of at least two of these surfaces (if there is more then one left). This intersection is a generic real-analytic Levi-flat submanifold of codimension 2 outside a complex analytic subvariety by Lemma 2.7. \square

3. UNIQUENESS PROPERTY FOR HOLOMORPHIC FUNCTIONS

For a generic submanifold M through the origin in \mathbb{C}^N , we wish to investigate when there exists a meromorphic function near the origin which is real valued on

M . By composing with a Möbius mapping of the real line onto the unit circle we see that this is equivalent to the existence of a meromorphic function which is unimodular on M , which in turn means that there are two relatively prime holomorphic functions f and g such that on M , $|f| = |g|$. We will thus define:

Definition 3.1. M has the *modulus uniqueness property* if $|f| = |g|$ on M , for holomorphic f and g defined in a neighbourhood of M , implies $f = cg$ for a unimodular constant c . We will say that M has the modulus uniqueness property at $p \in M$, if $M \cap U$ has the modulus uniqueness property for every connected neighbourhood U of $p \in M$.

In the following we will denote the local CR orbit at a point p by Orb_p . The motivation for our problem is the following theorem.

Theorem 3.2 (see [BER98]). *Let $M \subset U \subset \mathbb{C}^N$ be a generic real-analytic nowhere minimal submanifold of codimension d . Let $p \in M$ be such that Orb_p is of maximal dimension. Then there are coordinates $(z, w', w'') \in \mathbb{C}^n \times \mathbb{C}^{d-q} \times \mathbb{C}^q = \mathbb{C}^N$, where q denotes the codimension of Orb_p in M , vanishing at p such that near p , M is defined by*

$$\begin{aligned} \text{Im } w' &= \varphi(z, \bar{z}, \text{Re } w', \text{Re } w'') \\ \text{Im } w'' &= 0, \end{aligned}$$

where φ is a real valued real-analytic function with $\varphi(z, 0, s', s'') \equiv 0$. Moreover, the local CR orbit of the point $(z, w', w'') = (0, 0, s'')$, for $s'' \in \mathbb{R}^q$, is given by

$$\begin{aligned} \text{Im } w' &= \varphi(z, \bar{z}, \text{Re } w', s'') \\ w'' &= s''. \end{aligned}$$

So a natural question is to ask what happens at points where Orb_p is not of maximal dimension. In general there do not exist local normal coordinates such that $\text{Im } w'' = 0$ is one of the equations for M , but it is natural to ask when can we get a meromorphic function f such that $\text{Im } f = 0$ on M .

Before looking at this case we summarize the results for the easy cases.

Proposition 3.3. *Let M be a connected real-analytic CR submanifold through the origin. Then M does not have the modulus uniqueness property at the origin if any of the following holds,*

- (i) M is not generic,
- (ii) M is totally real,
- (iii) M is nowhere minimal and Orb_0 has the maximal dimension.

On the other hand M has the modulus uniqueness property at any point $p \in M$ if

- (iv) M is generic and minimal at some point.

Proof. The first three cases are clear. For the last one we just note that if M is minimal at some point, it is minimal on a dense open subset. If we had a nonconstant meromorphic function real valued on M , then on some small neighbourhood we would have that M is minimal and there would exist a holomorphic function with nonvanishing gradient which was real valued on M and this would give local foliation of M by smaller submanifolds of same CR dimension and this would violate minimality. \square

We also note that if M is not generic, but it is minimal, then M has the modulus uniqueness property inside the intrinsic complexification of M . So since Orb_p is always minimal then if we call \mathcal{X}_p the intrinsic complexification of Orb_p , then any meromorphic function real valued on M is constant in \mathcal{X}_p for any CR manifold.

It is then clearly useful to be able to construct \mathcal{X}_p and study its properties. The following constructions are described in [BER99]. We will look at a generic submanifold M defined in normal coordinates (z, w) in some neighbourhood U of the origin, and we will assume that U is small enough such that the defining equations for M complexify into $U \times {}^*U$, and we can take U to be connected. If $M = \{z \in U \mid r(z, \bar{z}) = 0\}$ we let $\mathcal{M} := \{(z, \zeta) \in U \times {}^*U \mid r(z, \zeta) = 0\}$. We define the *Segre manifolds* for $p \in U$

$$\begin{aligned} \mathfrak{S}_{2j+1}(p, U) := \{ & (z, \zeta^1, z^1, \dots, \zeta^j, z^j) \in U \times {}^*U \times U \times \dots \times {}^*U \times U \mid \\ & (z, \zeta^1), (z^1, \zeta^1), \dots, (z^j, \zeta^j), (z^1, \zeta^2), \dots, (z^{j-1}, \zeta^j), (z^j, \bar{p}) \in \mathcal{M} \} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{S}_{2j}(p, U) := \{ & (z, \zeta^1, z^1, \dots, z^{j-1}, \zeta^j) \in U \times {}^*U \times U \times \dots \times U \times {}^*U \mid \\ & (z, \zeta^1), (z^1, \zeta^1), \dots, (z^{j-1}, \zeta^{j-1}), (z^1, \zeta^2), \dots, (z^{j-1}, \zeta^j), (p, \zeta^j) \in \mathcal{M} \} \end{aligned}$$

where $\mathfrak{S}_1(p, U) = \{z \in U \mid (z, \bar{p}) \in \mathcal{M}\}$. If we define $\pi: \mathbb{C}^N \times \dots \times \mathbb{C}^N \rightarrow \mathbb{C}^N$ be the projection to the first coordinate, then we can define the *Segre sets* for $p \in U$ by $S_k(p, U) := \pi(\mathfrak{S}_k(p, U))$. Note that both $S_k(p, U)$ and $\mathfrak{S}_k(p, U)$ depend on both the point p and the neighbourhood U .

We have the following proposition, of which the first part is proved in [BER99] (Proposition 10.2.7), second part is then immediate.

Proposition 3.4. *For $k \geq 1$ we have*

$$S_k(p, U) = \bigcup_{q \in S_{k-1}(p, U)} S_1(q, U),$$

and if $k \geq 2$ we have

$$S_k(p, U) = \bigcup_{q \in S_{k-2}(p, U)} S_2(q, U).$$

Further, for normal coordinates where $U = U_z \times U_w$ we have the following (again proved in [BER99] as part of Proposition 10.4.1):

Proposition 3.5. *Let M be given by $w = Q(z, \bar{z}, \bar{w})$ in normal coordinates in U and let $p = (z^0, w^0)$. Then there exists an open set $V \subset {}^*U_z$ ($0 \in V$) such that $(z, w) \in U$ is in $S_2(p, U)$ if and only if there exists $\zeta \in V$ such that $w = Q(z, \zeta, \bar{Q}(\zeta, z^0, w^0))$.*

The above V is the set of all $\zeta \in {}^*U_z$ such that $\bar{Q}(\zeta, z^0, w^0) \in {}^*U_w$. In particular $0 \in V$. With this we prove the following useful lemma.

Lemma 3.6. *Suppose that $M \subset U \subset \mathbb{C}^N$ is a generic submanifold given by normal coordinates defined near the origin for a suitable U . Then for any point $p = (z^0, w^0) \in U$, the variety $\{(z, w) \in U \mid w = w^0\}$ is contained inside $S_2(p, U)$ (the second Segre set at p).*

Proof. Let M be given by $\{w = Q(z, \bar{z}, \bar{w})\}$ in normal coordinates. Thus $S_2(p, U) = \{(z, w) \mid w = Q(z, \zeta, \bar{Q}(\zeta, z^0, \bar{w}^0)), \zeta \in V\}$, where V is as in Lemma 3.5. In particular $0 \in V$ and thus since we are in normal coordinates, $Q(z, 0, w) \equiv Q(0, z, w) \equiv w$. Thus $\{(z, w) \mid w = w^0\} \subset S_2(p, U)$. \square

To be able to use this we note the following theorem given and proved in [BER99] (Theorems 10.5.2 and 10.5.4).

Theorem 3.7. *If M is as above then there exists a number j_0 such that for every sufficiently small neighbourhood U of $p \in M$, $S_{2j_0}(p, U)$ coincides with \mathcal{X}_p as germs at p , the intrinsic complexification of Orb_p .*

The number j_0 is called the Segre number of M at p , but we are only interested in the fact that such a number exists and not how it is arrived at. Another useful proposition from [BER99] (Proposition 10.2.28) is the following.

Proposition 3.8. *Let $p \in M \subset U$ and an integer $k_0 \geq 1$. Then there exist neighbourhoods $U'' \subset U' \subset U$ of p such that for all $q \in U''$, $\mathfrak{S}_k(q, U')$ is connected for all $k \leq k_0$.*

Next we assume that U is sufficiently nice (for example a polydisc).

Lemma 3.9. *Given $M \subset U$ in normal coordinates, then there is a small neighbourhood of the origin V such that for $p \in M \cap V$, \mathcal{X}_p contains $\{(z, w) \in U \mid w = w^0\}$ as germs at any $(z^0, w^0) \in \mathcal{X}_p$. If \mathcal{Z}_p is the intersection of all complex subvarieties of U which contain \mathcal{X}_p , then \mathcal{Z}_p contains $\{(z, w) \in U \mid w = w^0\}$ for any $(z^0, w^0) \in \mathcal{Z}_p$.*

Proof. Let M be in normal coordinates. We can always take U to be even smaller, so by Proposition 3.8 for a small enough neighbourhood of the origin U , there is a yet smaller neighbourhood of the origin V such that for $p \in V$, $\mathfrak{S}_k(p, U)$ is connected, for $k \leq 2(d+1) + 2$, d being the codimension of M . Note that the Segre number of M at any point is always less than or equal to $d+1$. By Theorem 3.7 we know $S_{2(d+1)}(p, W) = \mathcal{X}_p$ as germs for some small neighbourhood W of p . Hence $S_{2(d+1)+2}(p, W) = \mathcal{X}_p = S_{2(d+1)}(p, W)$ as germs at p . Let $k = 2(d+1)$. By Proposition 3.4, $S_{k+2}(p, U)$ is a union of $S_2(q, U)$ for $q \in S_k(p, U)$, and by Lemma 3.6 each $S_2(q, U)$ contains the set $\{(z, w) \mid w = w(q)\}$. In particular $S_{k+2}(p, U)$ contains the set $\{(z, w) \mid w = w(q)\}$ for each $q \in \mathcal{X}_p$ (for some small enough representative of the germ \mathcal{X}_p). Now we note that $\mathfrak{S}_{k+2}(p, W)$ is an open submanifold of $\mathfrak{S}_{k+2}(p, U)$, which is connected. We pull back the mapping $(z, w) \mapsto z$ to $\mathfrak{S}_{k+2}(p, U)$ and look at its rank to conclude that for $q \in S_{k+2}(p, W)$ we have $\{(z, w) \mid w = w(q)\} \subset S_{k+2}(p, W)$ as germs at q . This proves the first part.

To see the second part suppose that \mathcal{Z}_p did depend on z . Then we can intersect \mathcal{Z}_p with $\{(z, w) \mid z = z^0\}$ and the intersection must still contain \mathcal{X}_p projected on the w coordinate (it is of the form $\mathcal{X}_p = \mathbb{C}_z \times (\mathcal{X}_p)_w$). So we would get a different complex variety \mathcal{Z}'_p which contains \mathcal{X}_p . Intersection of \mathcal{Z}_p and \mathcal{Z}'_p would violate minimality of \mathcal{Z}_p . \square

Theorem 3.10. *Suppose that M is generic in normal coordinates. Suppose that f and g are two holomorphic functions such that $|f| = |g|$ on M . Then f/g depends only on w ; in other words, if h is a meromorphic function which is real valued on M , then h depends only on w .*

Proof. Obviously we only need to prove the first part as the second part follows. We can work in arbitrarily small neighbourhood U of the origin. As we noted before since Orb_p is minimal in \mathcal{X}_p we know that $f = cg$ in \mathcal{X}_p for any point p (where c depends on p of course). That is that the function f/g is constant on \mathcal{X}_p (if we take p outside the zero set of g). Since we know that as germs $\{(z, w) \mid w = w^0\} \subset \mathcal{X}_p$, then for any $1 \leq j \leq n$ we have $\frac{\partial}{\partial z_j}(f/g) = 0$ at p . Since M is generic and since $g = 0$ is a proper subvariety of M , then $\frac{\partial}{\partial z_j}(f/g) = 0$ holds for an open set of p in M , and then it holds for an open subset of U and thus for all of U . \square

4. SUBMANIFOLDS INSIDE LEVI-FLAT HYPERSURFACES

Since the question of the modulus uniqueness property of M (or alternatively of existence of a meromorphic function which is real valued on M) is the same as a question of M being contained in a certain kind of possibly singular real-analytic Levi-flat hypersurface, we can ask a weaker question; when is M contained in any possibly singular real-analytic Levi-flat hypersurface? We will consider M to be inside a hypersurface H if $M \subset \overline{H^*}$.

Proposition 4.1. *Suppose M is a connected generic real-analytic submanifold of codimension 2 in normal coordinates (z, w) and $M \subset \overline{H^*}$ where H is a irreducible possibly singular real-analytic Levi-flat hypersurface. Then in a possibly smaller neighbourhood of the origin, there exists a Levi-flat hypersurface \hat{H} defined by $\{(z, w) \mid \rho(w, \bar{w}) = 0\}$ such that $M \subset \overline{\hat{H}^*}$ as germs at 0. Furthermore, if M is not Levi-flat then $H = \hat{H}$ as germs at 0.*

Proof. If Orb_p is constantly of codimension 2 in M or constantly of codimension 1 in M , then by Theorem 3.2 we have a holomorphic function near the origin which is real valued on M and thus by Theorem 3.10 the defining equation for M already does not depend on z .

By Corollary 2.4, M cannot be minimal at any point. So suppose that M is not minimal and Orb_p is not of constant dimension. This means that it is not Levi-flat and thus by Corollary 2.3 it cannot be contained in $H_s \cap \overline{H^*}$ and thus must intersect H^* . This means that it must in fact intersect H^* on a dense open set in M (as H_s is contained in a proper subvariety). Suppose H is defined in U by $\{\rho(z, w, \bar{z}, \bar{w}) = 0\}$, in particular H is closed in U . Then for $p \in M \cap H^*$ we can see that $\mathcal{X}_p \subset H$, since in small enough neighbourhood of p , such as we have by Theorem 3.7, the k th Segre set of M is contained in the k th Segre set of H , and the Segre sets of H all lie in H for small enough neighbourhood of a nonsingular point of H . By Lemma 2.6, the Segre variety of H at p agrees with the Levi foliation of H at p , and since this (the Segre variety of H) is a proper subvariety of U , then if \mathcal{Z}_p is the smallest complex subvariety of U which contains \mathcal{X}_p , then $\mathcal{Z}_p \subset H$. This means in particular that $(\mathbb{C}_z \times pr_w(M \cap H^*)) \cap U \subset H$ (where pr_w is the projection onto second factor in the normal coordinates (z, w)), since \mathcal{Z}_p contains all the $(z, w) \in U$ for fixed w by Lemma 3.9. As H is closed and $M \cap H^*$ is dense in M , then $\mathbb{C}_z \times pr_w(M) \subset H$. Fix z^0 such that $\rho(z^0, w, \bar{z}^0, \bar{w}) = 0$ defines a hypersurface in \mathbb{C}_w , then this hypersurface is Levi-flat in \mathbb{C}_w . Define \hat{H} by $\{(z, w) \mid \rho(z^0, w, \bar{z}^0, \bar{w}) = 0\}$, this is Levi-flat again and further $M \subset \hat{H}$.

It is then clear that since $\mathcal{X}_p \subset H$, then $\mathcal{X}_p \subset \hat{H}$, thus near points p where Orb_p is of codimension 1 in M , they locally give a branch of a nonsingular Levi-flat hypersurface which must be contained in \hat{H} . Thus $M \subset \overline{\hat{H}^*}$.

If M is not Levi-flat then uniqueness of H comes from the fact that if M would be contained in two different Levi-flat hypersurfaces say H and H' it would be contained in their intersection and thus would be contained in the singular set of $H \cup H'$ and this is impossible by Corollary 2.3. \square

Our method of looking at projections onto the second factor of normal coordinates yields also first part of Theorem 1.1 which we can state as follows.

Theorem 4.2. *Let M be a germ of a generic real-analytic codimension 2 submanifold through the origin given in normal coordinates (z, w) and let M be nowhere minimal. Then $M \subset \overline{H^*}$, where H is a germ of a possibly singular real-analytic Levi-flat hypersurface if and only if the projection of M onto the second factor in (z, w) is contained in a germ of a possibly singular real-analytic hypersurface. Moreover, if M is not Levi-flat, then H is unique.*

Note that this theorem also gives a test for certain submanifolds being nowhere minimal. If we can compute a hypersurface containing the projection of M to the w coordinate, we need only check if it is Levi-flat or not.

Proof. The forward direction and uniqueness is proved by the preceding proposition. So suppose that $M \subset H$ where $H = \mathbb{C}_z \times H_w$ is a possibly singular hypersurface. We can assume that H is irreducible.

First suppose that Orb_0 is of maximal dimension. Then by Theorem 3.2 there exists (near 0) a holomorphic function real valued on M which thus defines a Levi-flat hypersurface (nonsingular one in fact). Also by Theorem 3.10 this function only depends on the w coordinate, this means that it really defines a Levi-flat hypersurface in \mathbb{C}^d (the w space) that contains $pr_w(M)$.

Next suppose that Orb_0 is not of maximal dimension. Fix a certain neighbourhood U where M is defined in the given normal coordinates. By Proposition 3.8 we can then pick a smaller $0 \in U' \subset U$ such that for all $p \in U'$, the Segre manifold $\mathfrak{S}_k(q, U)$ is connected. Making U' smaller we can assume it is of the form $U'_z \times U'_w$ where both U'_z and U'_w are polydiscs. We will pick a point $p \in M \cap U'$ where Orb_p is of maximal dimension (of codimension 1 in M).

By choosing above U small enough we can ensure that $pr_w(M)$ is subanalytic (see [BM88]). We look at a nonsingular point of this projection of highest dimension in $pr_w(M) \cap U'_w$. Obviously at this point $pr_w(M) \cap U'_w$ is either a hypersurface or codimension 2 since it is contained in H_w . If $pr_w(M)$ was a codimension 2 submanifold near some point, then it would be totally real, and thus M above it would be Levi-flat which is not the case. Thus there must be nonsingular points where $pr_w(M) \cap U'_w$ is a hypersurface. Further, since the \mathcal{X}_q really depends only on the w variables, it is clear that there is a point $p \in M \cap U'$, such that $pr_w(p)$ is a nonsingular point of $pr_w(M) \cap U'_w$, and such that Orb_p is of maximal dimension. Next, pick a small enough neighbourhood $V \subset U'$ of p , such that $pr_w(M \cap V)$ is a nonsingular hypersurface. Then $pr_w(M \cap V)$ agrees with one of the branches of H_w at $pr_w(p)$.

Locally in V (possibly taking smaller V) again we have a holomorphic function f in a neighbourhood of p that is real valued on M . We notice that in the proof of

Lemma 3.9 the only reason why we restrict to a smaller neighbourhood is so that we can apply Proposition 3.8, and hence we could have picked a neighbourhood of any point in U . So we see that in the proof of Theorem 3.10 we did not need to pick a neighbourhood of the origin, but we could have just used V as given above (possibly making it smaller). Hence f only depends on w , and thus again $\text{Im } f = 0$ defines a Levi-flat hypersurface near p which contains M near p . So in \mathbb{C}^d (the w coordinates) this hypersurface contains $pr_w(M \cap V)$ and thus agrees with a branch of H_w near $pr_w(p)$. By Lemma 2.1, H_w must be a Levi-flat hypersurface, and we are done. \square

5. ALMOST MINIMAL SUBMANIFOLDS

As we have already seen, if $M \subset \overline{H^*}$ and M is a generic nowhere minimal codimension 2 real-analytic submanifold, then at a point $p \in M \cap H^*$ where Orb_p is of codimension 1 in M , $\mathcal{X}_p \subset H^*$, that is, \mathcal{X}_p gives the Levi foliation of H . By Lemma 2.6, we have that locally the Segre variety Σ_p of H in U contains \mathcal{X}_p , and for $p \in H^*$, Σ_p is a proper analytic subvariety of U . So an obvious condition for M to be contained in a Levi-flat hypersurface is that \mathcal{X}_p is contained in a proper complex subvariety of U . Since \mathcal{X}_p is the smallest germ of a complex variety containing Orb_p , we let $\mathcal{Z}_p = \mathcal{Z}_{U,p}$ be the smallest complex subvariety of U that contains Orb_p (and thus \mathcal{X}_p).

Definition 5.1. Let $M \subset U \subset \mathbb{C}^N$ be a generic submanifold. We will say that M is *almost minimal in U* , if there exists a point p such that $\mathcal{Z}_{U,p}$ contains an open set, and we will say that p makes M almost minimal in U . We'll say that a generic submanifold M is *almost minimal at p* , if it is almost minimal in every neighbourhood of p .

If M is minimal at $p \in U$, then it is, of course, almost minimal in U . And if a connected M is real-analytic and minimal at one point, it is minimal on an open dense set, and thus it is almost minimal at every point.

An example of a nowhere minimal submanifold that is almost minimal is the M_λ family given in the introduction for λ irrational. See §8 for this example worked out. It should be noted that if M is nowhere minimal, then the points where it is almost minimal are contained in a proper real analytic subvariety in M . This is because if M is almost minimal at p and nowhere minimal, then Orb_p must not be of maximal dimension.

Theorem 5.2. *Suppose that $M \subset \mathbb{C}^N$ is a germ of a real-analytic generic submanifold of codimension 2 through 0, and suppose $M \subset \overline{H^*}$ where H is a germ of a possibly singular real-analytic Levi-flat hypersurface. Then M is not almost minimal at 0.*

Proof. Let U be a small enough connected neighbourhood of the origin such that both M and H are closed in U and further such that their defining equations are complexifiable in U . M cannot be minimal at any point by Theorem 2.4. Further, if M is Levi-flat then Orb_p is constantly of codimension 2 in M . This means that Orb_p is in fact complex analytic and is contained in the Segre variety (the first Segre set of M in U) and thus cannot be almost minimal.

So suppose on a dense open set of points of M , Orb_p is of codimension 1 in M , and in fact, if p makes M almost minimal in U then Orb_p has to be of codimension

1 in M . Further, $M \cap H^*$ is non-empty (since M is not Levi-flat) and as noted before is thus open and dense in M . Also as noted above, the p that makes M almost minimal cannot lie in $M \cap H^*$.

So pick a small neighbourhood of any $p \in M \cap H_s$ where Orb_p is of codimension 1 in M . Then by Theorem 3.2, there is a small neighbourhood V of p on which there exist normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^2$ vanishing at p , such that M is given by $\text{Im } w_1 = \rho(z, \bar{z}, \text{Re } w)$ and $\text{Im } w_2 = 0$, and further, that the \mathcal{X}_q are then given by $w_2 = s$ (we'll denote this set as $\{w_2 = s\}$) for some $s \in (-\epsilon, \epsilon)$. We can take V to be a polydisc in the (z, w) coordinates. If $M \cap \{w_2 = s\}$ (which is the CR orbit) contains a point which is in H^* , then as we reasoned above $\{w_2 = s\} \subset H$ since it agrees with the Levi foliation of H at some point in H^* . As $M \cap H^*$ is dense in M , then $\{w_2 = s\} \subset H$ for all $s \in (-\epsilon, \epsilon)$. This means that in V , $\text{Im } w_2$ divides the defining function of H in U . Thus the Segre variety of H in U contains the Segre variety of $\{\text{Im } w_2 = 0\}$ at all points in $\{\text{Im } w_2 = 0\}$. We wish to show that Orb_p is contained in a proper complex analytic subvariety. Either it is contained in a nondegenerate Segre subvariety of H in U or the Segre variety of H in U is degenerate at all points of $\text{Orb}_p = M \cap \{w_2 = 0\}$, but the set of points where the Segre variety of H is degenerate is a proper analytic subset as we remarked before. In any case p does not make M almost minimal in U , and thus M is not almost minimal in U . \square

Corollary 5.3. *Suppose that $M \subset \mathbb{C}^N$ is a connected real-analytic generic submanifold of codimension 2 through 0, and M is almost minimal at 0. Then M has the modulus uniqueness property at 0.*

6. INFINITESIMAL CR AUTOMORPHISMS

We will now look at the dimension of $\text{hol}(M, p)$, the space of infinitesimal holomorphisms at p (see the introduction for terminology) if M is almost minimal at p . As motivation we have the following theorem.

Theorem 6.1 (Baouendi-Ebenfelt-Rothschild see [BER98]). *Let $M \subset \mathbb{C}^N$ be a connected real-analytic CR submanifold that is holomorphically nondegenerate. If M is minimal at any point $p \in M$, then $\dim_{\mathbb{R}} \text{hol}(M, q) < \infty$ for all $q \in M$. If M is nowhere minimal then $\dim_{\mathbb{R}} \text{hol}(M, q) = 0$ or $\dim_{\mathbb{R}} \text{hol}(M, q) = \infty$ for q in a dense open subset of M .*

Thus it remains to see at exactly what points $\text{hol}(M, q)$ is finite dimensional in case M is nowhere minimal. Our main result of this section is that it turns out that the points where M is almost minimal are such points. We restate Theorem 1.2 from the introduction for convenience.

Theorem. *Let $M \subset \mathbb{C}^N$ be a connected, real-analytic holomorphically nondegenerate generic submanifold and suppose $p \in M$ and M is almost minimal at p . Then*

$$\dim_{\mathbb{R}} \text{hol}(M, p) < \infty.$$

The proof is essentially the same as in [BER98] or [BER99] for minimal submanifolds, although we will require Lemma 6.3 to modify that proof. It would not be needed, if we had a more general way of showing that certain CR orbits (of the highest dimension for example) were holomorphically nondegenerate whenever M was.

However this is not so. For example, the manifold defined in $(z_1, z_2, w_1, w_2) \in \mathbb{C}^4$, by

$$\begin{aligned} \operatorname{Im} w_1 &= |z_1|^2 + (\operatorname{Re} w_2) |z_2|^2, \\ \operatorname{Im} w_2 &= 0, \end{aligned}$$

is holomorphically nondegenerate. However, the CR orbit at 0 is defined by $\operatorname{Im} w_1 = |z_1|^2$ and $w_2 = 0$, and so $\frac{\partial}{\partial z_2}$ is a holomorphic vector field tangent to it. We can, however, prove the following result for almost minimal submanifolds.

Lemma 6.2. *Suppose $M \subset U$ is a holomorphically nondegenerate generic submanifold, and $p \in M$ is such that $\mathcal{Z}_{U,p} = U$, that is, p makes M almost minimal in U . Then Orb_p is holomorphically nondegenerate.*

The proof is essentially contained the proof of Theorem 1.2 below, and uses the following technical lemma.

Lemma 6.3. *Let $M \subset U_z \times U_w \subset \mathbb{C}^N$ be a generic submanifold given in normal coordinates (z, w) in a sufficiently small $U = U_z \times U_w$ by $w = Q(z, \bar{z}, \bar{w})$. Suppose there exists a holomorphic function $f: \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \rightarrow \mathbb{C}$ defined in a neighbourhood of the origin such that $f(z, \bar{z}, w)$ is defined in U , and there exists a point $p \in M$ and $f(z, \bar{z}, w) = 0$ on Orb_p . Then there exists a holomorphic function $g: U_w \subset \mathbb{C}^d \rightarrow \mathbb{C}$ such that $g(w) = 0$ on Orb_p .*

Proof. First note that Lemma 3.9 implies that locally near p , we can find a germ of a holomorphic function φ such that $\varphi(w) = 0$ defines \mathcal{X}_p . Thus we can do a local change of coordinates in w only, setting $w' = \psi(w)$ and $w'' = \varphi(w)$ for some function ψ . So locally we have Orb_p defined in the coordinates z, w', w'' (which are no longer normal coordinates) by $w' = \tilde{Q}(z, \bar{z}, \bar{w}')$ and $w'' = 0$, for some function \tilde{Q} defined in a neighbourhood of p . We can now also write f in the z, w', w'' coordinates by abuse of notation as $f(z, \bar{z}, w', w'')$. Assuming U is small enough and the neighbourhood where w', w'' are defined is also small enough we can define a complexified version of Orb_p by setting $\bar{w}' = \xi$ and $\bar{z} = \zeta$ by $\xi = \tilde{Q}(\zeta, z, w')$ and call this \mathcal{C} . Since $f(z, \bar{z}, w', 0) = 0$ on Orb_p , then as Orb_p is maximally real in \mathcal{C} we have that $f(z, \zeta, w', 0) = 0$ on \mathcal{C} and as z, ζ, w' are free variables on \mathcal{C} we know that $f(z, \bar{z}, w', w'') = 0$ when $w'' = 0$, but $w'' = 0$ defines \mathcal{X}_p , so f is identically zero on all of \mathcal{X}_p . Since \mathcal{X}_p is defined by an equation which is independent of z , then if we fix z^0 where $(z^0, w^0) = p \in M$, and we take $g(w) := f(z^0, \bar{z}^0, w)$, then $g(w)$ as a function of (z, w) but independent of z is zero on \mathcal{X}_p and thus on Orb_p . And g is defined in all of U_w and thus we are done. \square

We need to characterize $\operatorname{hol}(M, p)$ in a more natural way for the proof and the following proposition is proved in [BER99] (Proposition 12.4.22).

Proposition 6.4. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold and X a germ of a real, real-analytic vector field on M . Then $X \in \operatorname{hol}(M, p)$ if and only if there exists a germ \mathcal{X} at p of a holomorphic vector field in \mathbb{C}^N such that $\operatorname{Re} \mathcal{X}$ is tangent to M and $X = \operatorname{Re} \mathcal{X}|_M$.*

It is not hard to see that if \mathcal{X} is a holomorphic vector field as above and $\tilde{\varphi}(z, \tau)$ is the holomorphic flow of \mathcal{X} and $X = \operatorname{Re} \mathcal{X}$ and $\varphi(z, \bar{z}, t)$ is the flow of X , then φ and $\tilde{\varphi}$ coincide when $t = \tau \in \mathbb{R}$.

We will need the notion of k -nondegeneracy, but instead of giving the definition of being k -nondegenerate at a point, we can just take the following proposition from [BER99] (Corollary 11.2.14) and treat it as a definition.

Proposition 6.5. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of codimension d and CR dimension n given in normal coordinates $Z = (z, w) \subset U \subset \mathbb{C}^n \times \mathbb{C}^d$ by $w = Q(z, \bar{z}, \bar{w})$. Then M is k -degenerate at $p = (z_p, w_p)$ (sufficiently close to 0) if and only if*

$$\text{span} \left\{ \left(\frac{\partial}{\partial \bar{z}} \right)^\alpha \frac{\partial \bar{Q}_j}{\partial Z}(\bar{z}_p, z_p, w_p) \mid j = 1, \dots, d, 0 \leq |\alpha| \leq k \right\} = \mathbb{C}^N.$$

We must prove a result about finite jet determination of biholomorphisms of almost minimal submanifolds, which may be of interest on its own. As before let $\mathcal{Z}_{U,p}$ be the smallest complex analytic variety containing Orb_p . So if M is almost minimal in U and p is the point that makes it almost minimal then we have the following proposition.

Proposition 6.6. *Let $M, M' \subset \mathbb{C}^N$ be real-analytic generic submanifolds of codimension d defined in open sets U and U' respectively. Let f and g be two holomorphic mappings taking U to U' and M to M' . Let $p \in M$ be such that $\mathcal{Z}_{U,p} = U$ and suppose M is k_0 -nondegenerate at p . Also suppose that $f(p) = g(p) = p'$, $f_*(T_p^c M) = T_{p'}^c M$ and $g_*(T_p^c M) = T_{p'}^c M$. Then if $j_p^{(d+1)k_0} f = j_p^{(d+1)k_0} g$ then $f = g$.*

Proof. The proposition follows from Corollary 12.3.8 in [BER99] which is a slightly stronger result than the above, but which says that $f = g$ in Orb_p only. As $\mathcal{Z}_{U,p} = U$, then of course $f = g$ everywhere on U . \square

To be able to use Proposition 6.6 we need to know that M is k -nondegenerate at the right points. From [BER99] we have the following lemma (part of Theorem 11.5.1).

Lemma 6.7. *Suppose $M \subset \mathbb{C}^N$ is a connected real-analytic generic submanifold of CR dimension n that is holomorphically nondegenerate. Then there exists a proper real-analytic subvariety $V \subset M$ such that M is ℓ -nondegenerate for all $p \in M \setminus V$ for some $1 \leq \ell \leq n$.*

The $\ell = \ell(M)$ is the *Levi-number* of M .

Proof of Theorem 1.2. First suppose that $X^1, \dots, X^m \in \text{hol}(M, p)$ are linearly independent over \mathbb{R} . Suppose that $x = (x_1, \dots, x_r)$ be local coordinates for M vanishing at p . Here we may write $X^j = \sum_{k=1}^r X_k^j(x) \frac{\partial}{\partial x_k}$, or for short $X^j \cdot \frac{\partial}{\partial x}$. We let $y \in \mathbb{R}^m$ and denote by $\varphi(t, x, y)$ the flow of the vector field $y_1 X^1 + \dots + y_m X^m$, that is the solution of

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(t, x, y) &= \sum_{j=1}^m y_j X^j(\varphi(t, x, y)), \\ \varphi(0, x, y) &= x. \end{aligned}$$

Since $\varphi(st, x, y) = \varphi(t, x, sy)$ (which follows from the uniqueness of the solution), we can choose $\delta > 0$ small enough such that there exists $c > 0$ such that the flow

is smooth for (t, x, y) where $|t| \leq 2$, $|x| \leq c$ and $|y| \leq \delta$. We look at the time-one map denoted by

$$F(x, y) := \varphi(1, x, y).$$

We have the following lemma proved in [BER98] and [BER99] (Lemma 12.5.10).

Lemma 6.8. *Let F , x , c , and δ be as above. There exists $\gamma > 0$ such that $\gamma < \delta$ such that for any fixed $y^1, y^2 \in \mathbb{R}^m$ where $|y^j| \leq \gamma$, $j = 1, 2$, if $F(x, y^1) \equiv F(x, y^2)$ for $|x| \leq c$ then necessarily $y^1 = y^2$.*

Suppose that X^j are as above and are in $\text{hol}(M, p)$. Denote by $V \subset M$ the neighbourhood of p given by $|x| < c$ where x and c are as above. Let $\gamma > 0$ be picked as in Lemma 6.8. From Proposition 6.4 (and discussion afterward) it follows that for a fixed y such that $|y| < \gamma$, there exists a biholomorphism $z \mapsto \tilde{F}(z, y)$ defined in some connected open neighbourhood $U \subset \mathbb{C}^N$ of $V \subset M$, taking M into M . We can take γ smaller if necessary. Further, if $z(x)$ is the parametrization of M near p , these satisfy $F(x, y) = \tilde{F}(z(x), y)$, where F is the time-one map defined above.

As M is holomorphically nondegenerate, then by Lemma 6.7 we have that outside a real-analytic set it is ℓ -nondegenerate. Note that by Proposition 6.5 we have that this set is actually contained in a set defined by the vanishing of a function of the form $\varphi(z, \bar{z}, w)$, that is a real-analytic function in z , but holomorphic in w . Since M is almost minimal at p and if $Z_{U,q} = U$, then $Z_{U,q'} = U$ for all $q' \in \text{Orb}_q$, we know by Lemma 6.3 that there must exist a $q \in M$ such that $Z_{U,q} = U$ and M is ℓ -nondegenerate at q .

Thus we have satisfied requirements of Proposition 6.6, and by applying Lemma 6.8 we see that we have an injective mapping

$$y \mapsto j_q^{(d+1)\ell} \tilde{F}(\cdot, y) \in J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q,$$

where $J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q$ is the jet space at q of germs of holomorphic mappings from \mathbb{C}^N to \mathbb{C}^N . As $J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q$ is finite dimensional, then obviously $m \leq \dim_{\mathbb{R}} J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q$. Thus $\dim_{\mathbb{R}} \text{hol}(M, p) \leq \dim_{\mathbb{R}} J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q$. \square

7. ALGEBRAIC SUBMANIFOLDS

A manifold is real-algebraic if it is contained in a real-algebraic variety of the same dimension. The following theorem is basically proved in [BER99] (Theorem 13.1.10). It is also easily seen as a direct consequence of Tarski-Seidenberg (see [BM88]) and of the Chevalley theorem (see for example [Loj91]). That is, projections of real or complex algebraic varieties are either semi-algebraic (in the real case) or constructible (in the complex case) but in both cases they are contained in a real or complex algebraic variety of the same dimension. And since \mathcal{X}_p is locally given as projection of a Segre manifold which is complex-algebraic if M is real-algebraic, we have the following.

Theorem 7.1. *Let M be a real-algebraic generic submanifold and $p \in M$. Then Orb_p is real-algebraic and similarly \mathcal{X}_p is contained in a complex algebraic variety of the same dimension.*

If M is nowhere minimal and real-algebraic, U is an open set, and $p \in M \cap U$, then Orb_p is contained in a proper complex analytic subvariety of U . So we have the following corollary.

Corollary 7.2. *Suppose $M \subset \mathbb{C}^N$ is a connected real-algebraic generic submanifold. Then M is almost minimal at $p \in M$ if and only if M is minimal at some point.*

Corollary 7.3. *Let M be nowhere minimal real-analytic generic submanifold which is almost minimal at $p \in M$. Then M is not biholomorphic to a real-algebraic generic submanifold.*

This is because almost minimality would be preserved under biholomorphisms. The M_λ for λ irrational defined in the introduction is therefore an example of a submanifold not biholomorphic to a real-algebraic one.

We now prove the second part of Theorem 1.1 which we can state as follows.

Theorem 7.4. *Let M be a germ of a real-algebraic nowhere minimal generic submanifold of codimension 2. Then there exists a germ of a Levi-flat real-algebraic singular hypersurface H such that $M \subset \overline{H^*}$. Moreover, if M is not Levi-flat, then H is unique.*

Proof. If Orb_p is of constant codimension 2 in M , then we note that since normal coordinates are obtained by implicit function theorem and there exists an algebraic implicit function theorem, then we can find algebraic normal coordinates where M is given by $w = Q(z, \bar{z}, \bar{w})$. See [BER99] for the construction of the normal coordinates. Since Orb_p is of constant dimension 2 in M , it agrees locally with its intrinsic complexification which is then given by keeping w constant. Thus the vector fields $\frac{\partial}{\partial z_k}$ and $\frac{\partial}{\partial \bar{z}_k}$ for all $1 \leq k \leq n$ annihilate the defining equations for M (on M and since M is generic, in a neighbourhood). Thus M is given by $w_1 = Q_1(\bar{w}_1, \bar{w}_2)$ and $w_2 = Q_2(\bar{w}_1, \bar{w}_2)$. From this we can easily construct two algebraic holomorphic functions which are real valued on M , and we are done.

So assume that Orb_p is of codimension 1 in M on an open and dense set. Fix a certain representative of the germ of M . Pick a point $p \in M$ near the origin where Orb_p is of constant dimension 1 in M . Let U be a suitable neighbourhood of p . And let $p \in U' \subset U$ be a smaller neighbourhood such that if $\mathfrak{S}_k(q, U)$ is the k th Segre manifold at $q \in U'$, $\mathfrak{S}_k(q, U)$ is connected. We'll call \mathcal{U} the ambient space of $\mathfrak{S}_k(q, U)$, that is the $U \times {}^*U \times U \times \dots \times {}^*U \times U$ or $U \times {}^*U \times U \times \dots \times U \times {}^*U$ depending on whether k is even or odd. Then again denote by $\pi: \mathbb{C}^N \times \dots \times \mathbb{C}^N \rightarrow \mathbb{C}^N$ the projection onto the first factor, but we will define π on the space $\mathcal{U} \times U'$. Then define

$$\mathfrak{S}_k(M \cap U', U) := \{(\chi, q) \in \mathcal{U} \times U' \mid \chi \in \mathfrak{S}_k(q, U), q \in M \cap U'\}.$$

$\mathfrak{S}_k(M \cap U', U)$ is a real-algebraic set in $\mathcal{U} \times U'$ and thus $\pi(\mathfrak{S}_k(M \cap U', U))$ is semi-algebraic by Tarski-Seidenberg. We know that if U is small enough and k is large enough then \mathcal{X}_q will lie in $\pi(\mathfrak{S}_k(M \cap U', U))$, and further they form a nonsingular Levi-flat hypersurface at that point. Since a semialgebraic set is contained in an algebraic set of the same dimension, that is, there exists a polynomial p defining a hypersurface $H = \{\xi \in \mathbb{C}^N \mid p(\xi, \bar{\xi}) = 0\}$ that contains $\pi(\mathfrak{S}_k(M \cap U', U))$. Since $\pi(\mathfrak{S}_k(M \cap U', U))$ locally agrees with a nonsingular Levi-flat hypersurface we can take H to be irreducible. Then H is Levi-flat at p and by Lemma 2.1 it is Levi-flat.

As germs at p we can see that $M \subset \overline{H^*}$. Further, since this happens at every point where Orb_p is of codimension 1 in M , and these are open and dense in M , then this must happen in some neighbourhood of the origin and hence as germs at the origin. Uniqueness was proved previously already in §4. \square

8. EXAMPLE

Let M_λ , $\lambda \in \mathbb{R}$, be the generic, nowhere minimal submanifold of \mathbb{C}^3 , with holomorphic coordinates (z, w_1, w_2) defined by

$$\begin{aligned}\bar{w}_1 &= e^{iz\bar{z}}w_1, \\ \bar{w}_2 &= e^{i\lambda z\bar{z}}w_2.\end{aligned}$$

We wish to classify the λ 's for which M_λ has the modulus uniqueness property at the origin. That is, we will wish to find out when there exists a nontrivial meromorphic function which is real valued on M_λ . Note that we can always find a multivalued function which is real valued on M_λ , and that is

$$(z, w_1, w_2) \mapsto \frac{w_1^\lambda}{w_2}.$$

In fact, this proves that M_λ is nowhere minimal. Further, if λ is rational, say $\lambda = a/b$, then $(z, w_1, w_2) \mapsto w_1^a/w_2^b$ is a meromorphic function that is real valued on M . Thus M_λ does not have the modulus uniqueness property, and further, since it is of codimension 2, it is not almost minimal at the origin.

Let's check that M_λ is almost minimal at 0 when λ is irrational. For this we need to compute the Segre sets. We can compute the third Segre set at (z^0, w_1^0, w_2^0) , where $w_1^0 \neq 0$ and $w_2^0 \neq 0$, by the following mapping (see [BER99])

$$(t_1, t_2, t_3) \mapsto (t_3, \overline{w_1^0} e^{i(t_3 t_2 - t_2 t_1 + t_1 \overline{z^0})}, \overline{w_2^0} e^{i\lambda(t_3 t_2 - t_2 t_1 + t_1 \overline{z^0})}).$$

We can pick t_3 to be anything we want, and we can pick t_2 and t_1 such that the second component is anything we want since w_1^0 is non zero. By adding multiples of 2π , we can add a dense set of rotations of the third component because λ is irrational. This means, that the closure of this set will be 5 dimensional, and thus we will not be able to fit it inside a proper complex analytic subset and so M_λ is almost minimal.

We give an alternative more direct proof that M_λ does not have the modulus uniqueness property at the origin, and in fact prove a slightly more general theorem that can be used for generating further examples.

Proposition 8.1. *Suppose that M is a real-analytic, generic submanifold of codimension d inside \mathbb{C}^{n+d} passing through the origin that can be defined by normal coordinates of the form*

$$w_j = Q_j(z, \bar{z})\bar{w}_j$$

and further suppose that for any integer K the functions $Q_1^{k_1} \cdot Q_2^{k_2} \cdot \dots \cdot Q_d^{k_d}$ for $0 \leq k_1, \dots, k_d \leq K$ are linearly independent as functions. Then there does not exist a non-constant meromorphic (nor a holomorphic) function h defined in a neighbourhood of 0 which is real valued on M .

Proof. For easier notation we will assume $n = 1$ and $d = 2$. So suppose that $h = f/g$ is real valued on M , meaning that on M we have $f\bar{g} - \bar{f}g = 0$. We have proved before that h does not depend on z . Suppose that f and g are defined by

Taylor series expansions about 0. Thus

$$\begin{aligned} f(w_1, w_2) &= \sum_{k,l \geq 0} f_{kl} w_1^k w_2^l, \\ g(w_1, w_2) &= \sum_{n,p \geq 0} g_{np} w_1^n w_2^p. \end{aligned}$$

On M we therefore have (as $\bar{w}_i = \bar{Q}_i w_i$)

$$\begin{aligned} 0 &= f\bar{g} - \bar{f}g \\ &= \left(\sum_{k,l,n,p \geq 0} f_{kl} \bar{g}_{np} w_1^k w_2^l \bar{w}_1^n \bar{w}_2^p \right) - \left(\sum_{k,l,n,p \geq 0} \bar{f}_{kl} g_{np} w_1^n w_2^p \bar{w}_1^k \bar{w}_2^l \right) \\ &= \sum_{s,t \geq 0} \left(\sum_{k+n=s, l+p=t} f_{kl} \bar{g}_{np} (\bar{Q}_1)^p (\bar{Q}_2)^n - \bar{f}_{kl} g_{np} (\bar{Q}_1)^l (\bar{Q}_2)^k \right) w_1^s w_2^t. \end{aligned}$$

For a fixed z then we have a holomorphic function in w_1 and w_2 that is 0 on a generic manifold (restriction of M_λ to the (w_1, w_2) space) and is thus identically zero. This means that each coefficient is 0, and since by assumption these are linear combinations of powers of Q_1 and Q_2 , we get

$$f_{(s-k)(t-l)} \bar{g}_{kl} - \bar{f}_{kl} g_{(s-k)(t-l)} = 0.$$

The above is true for all $s, t \geq 0$ and all $k \leq s, l \leq t$. This implies that either $g \equiv 0$ or that $f_{kl} = C g_{kl}$ for all k, l for some constant C . Meaning there is no nonconstant meromorphic function which is real valued on M . \square

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